SMOOTH MANIFOLDS FALL 2022 - HOMEWORK 9

Problem 1. Show that if M_i are compact, connected, oriented manifolds of the same dimension for $i = 1, 2, 3$, and $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are C^∞ functions, then $\deg(g \circ f) = \deg(g)\deg(f)$.

Problem 2. Show that if M and N are compact, connected and oriented manifolds of the same dimension, and $F : M \to N$ has $\deg(F) = 0$, then F has a critical point.

Problem 3. Let A be an $n \times n$ square matrix with integer entries, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and $F_A : \mathbb{T}^n \to \mathbb{T}^n$ be defined by

$$F_A([x]) = [Ax]$$

where the notation $[x]$ denotes the equivalence class $x + \mathbb{Z}^n$.

(1) Show that F_A is well-defined.
(2) Compute $\deg(F_A)$ by computing the signed number of preimages of a regular value of F_A.
(3) Compute $\deg(F_A)$ by computing $\int F_A^* \omega$, where ω is the standard n-form $dx_1 \wedge \cdots \wedge dx_n$.

Non-graded.

Problem 4. Let Γ be a countable group acting properly discontinuously on an oriented manifold M by diffeomorphisms. Show that the quotient manifold M/Γ is orientable if and only if every $\gamma \in \Gamma$ is orientation-preserving.

Problem 5. Let ω be a top form on a C^∞ manifold M and $\varphi_t : M \to M$ be a flow on M generated by a vector field X. Show that $\varphi_t^*\omega = \omega$ for all $t \in \mathbb{R}$ if and only if $\iota_X \omega$ is closed. Use this to find a condition for a flow generated by a vector field $X = \sum f_i \frac{\partial}{\partial x_i}$ to preserve the standard top form $\omega = dx_1 \wedge \cdots \wedge dx_n$ on $\mathbb{R}^n$.

Problem 6. Let $f : M \to M$ be an orientation-preserving diffeomorphism of an oriented compact manifold M. Fix a non-vanishing top form ω_0. Show that there is a unique C^∞ function $\lambda : M \to \mathbb{R}$ such that $f^*\omega = e^\lambda \omega$, and f preserves *some* non-vanishing top form if and only if there exists a C^∞ function $h : M \to \mathbb{R}$ such that

$$\lambda = h \circ f - h.$$

Problem 7. If G is a connected, compact Lie group, consider the squaring map $s(g) = g^2$.

(a) Show that if $G = \mathbb{T}^n$, then $\deg(s) = 2^s$. Check this formula both by computing the degree at a regular value, as well as using a volume form.
(b) Show that if G is the set of unit quaternions, then $\deg(s) = 2$ [*Hint*: Show that if $g \neq \pm 1$, then there is a unique 1-parameter subgroup passing through g and that any square root must belong to it]

Remark 1. If you know about the structure of compact Lie groups, you may try the following more difficult exercise: $\deg(s) = 2^r$, where r is the maximal dimension of a connected abelian subgroup of G.